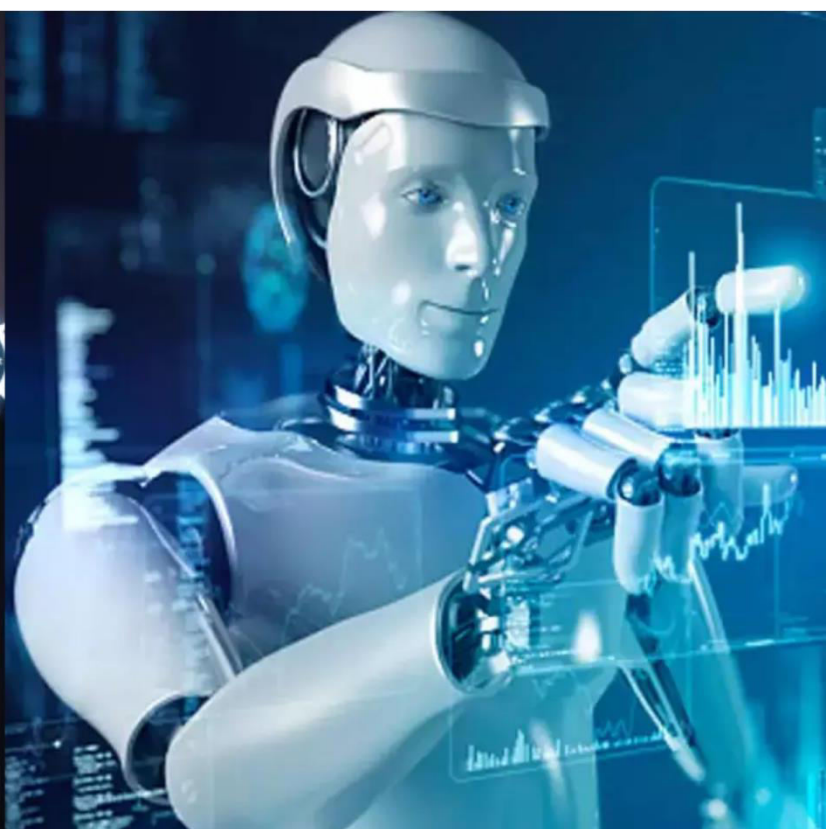




International Journal of Innovative Research in Science Engineering and Technology (IJIRSET)

(A Monthly, Peer Reviewed, Refereed, Scholarly Indexed, Open Access Journal)



Impact Factor: 8.699

Volume 15, Issue 4, April 2026

Simultaneous Approximation by a Certain Summation-Integral type Operator

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ABSTRACT: The aim of this paper is to investigate the simultaneous approximation properties of the Baskakov–Durrmeyer type operator. In particular, we study the approximation of functions and their derivatives simultaneously by the operator $G_\sigma(f; w)$. Within the framework of simultaneous approximation, we establish several direct results concerning the approximation behavior of the operator. These results provide further insight into its convergence and smoothness properties.

KEYWORDS: Baskakov-Durrmeyer Operator, Rate of Convergence, Asymptotic Approximation, Simultaneous Approximation.

I. INTRODUCTION

Durrmeyer [1] introduced integral modifications of Bernstein polynomials, leading to more adaptable approximation tools. This initiated significant research on integral type operators, with many researchers contributing to our understanding of their properties, limitations, and applications (we refer to [2] [3] [4] [5] [6] [7] [8] [9] [10] [11]). Gupta (1) introduced a novel modification to the Baskakov operator, a fundamental tool in approximation theory. Gupta's approach involved incorporating the weight function of the Beta operator, thereby enabling the approximation of Lebesgue integrable mappings over the interval $[0, \infty)$ as

$$(1) \quad v_n(f; w) = \sum_{k=1}^{\infty} \binom{n+l-1}{l} \frac{w^l}{(1+w)^{n+l}} \int_0^{\infty} v_{n,l}(y) f(y) dy, \quad w \in [0, \infty),$$

where $v_{n,l}(y) = \frac{1}{B(l+1, n)} \frac{w^k}{(1+w)^{n+l+1}}$ and $B(l+1, n)$ is Euler beta function.

This modification not only broadened the scope of the Baskakov operator but also showcased the integration of diverse mathematical concepts to enhance approximation techniques.

We denote $C_\beta[0, \infty)$, $(\beta > 0)$ be the class of continuous mappings on $[0, \infty)$ satisfying $|f(y)| \leq My^\beta$, $M > 0$. with norm

$$\|f\|_\beta = \sup_{0 < y < \infty} |f(y)| y^{-\beta}.$$

We introduce the operator $G_\sigma(f; w)$ such that,

$$(2) \quad G_\sigma(f; w) = (\sigma + 1) \sum_{l=1}^{\infty} v_{\sigma+2, l-1}(w) \int_0^{\infty} v_{\sigma+2, l-1}(y) f(y) dy, \quad w \in [0, \infty),$$

where

$$v_{\sigma, l}(w) = \binom{\sigma+l-1}{l} \frac{w^l}{(1+w)^{\sigma+l}}.$$

aimed at improving the approximation (space of Lebesgue integrable functions on $[0, \infty)$).

Assuming that $R_\sigma(w, y)$ is the kernel of equation (2) in the form expressed by

$$(3) \quad G_\sigma(f; w) = \int_0^\infty R_\sigma(w, y) f(y) dy,$$

such that

$$R_n(w, y) = (\sigma + 1) \sum_{l=1}^{\infty} v_{\sigma+2, l-1}(w) v_{\sigma+2, l-1}(y).$$

To ensure the study is self-contained, we address the importance of the Ditzian-Totik moduli of smoothness and unified K –functional [12]. We study the possibility of approximation of derivatives of functions by corresponding derivatives of sequence of operators. So we will prove that convergence is obtained for sufficiently smooth function we shall also find rate of approximation.

This study is known as simultaneous approximation behaviour of operators sequence. The direct results in simultaneous approximation are obtained in local setting. For global results, we use the properties of Steklov means to obtain error estimates in terms of modulus of smoothness. The Steklov averaging means are defined as follows. Let $f \in C[a, b]$ and $[a_1, b_1] \subset (a, b)$. Then, for sufficiently small $\lambda > 0$, the Steklov mean $f_{\lambda, s}$ of s -th order corresponding to f is defined by

$$f_{\lambda, s}(t) = \lambda^{-s} \int_{-\lambda/2}^{\lambda/2} \dots \int_{-\lambda/2}^{\lambda/2} \left(f(t) + (-1)^{s-1} \Delta_{\sum_{i=1}^s t_i}^s f(t) \right) \prod_{i=1}^s dt_i, t \in [a_1, b_1].$$

Lemma 1. For the function $f_{\lambda, s}$, we have

- $f_{\lambda, s}$ has derivatives up to order s over $[a_1, b_1]$;
- $\|f_{\lambda, s}^{(r)}\|_{[a_1, b_1]} \leq C \lambda^{-r} \omega_s(f, \lambda, [a, b]), r = 1, 2, \dots, s$;
- $\|f - f_{\lambda, s}\|_{[a_1, b_1]} \leq C \omega_s(f, \lambda, [a, b])$;
- $\|f_{\lambda, s}\|_{[a_1, b_1]} \leq C \|f\|_{[a, b]}$;
- $\|f_{\lambda, s}^{(s)}\|_{[a_1, b_1]} \leq C \lambda^{-s} \|f\|_{[a, b]}$,

where C is a certain constant not same at the each occurrence that depends on s but is independent of f and λ .

Proof. Following [13] or [14], the proof follows easily.

II. PRELIMINARIES

To establish the primary conclusions of this study, a few introductory lemmas are proved. Let e_i be the function $e_i(w) = w^i, i - 1 \in \mathbb{N}, w \in [0, \infty)$.

Lemma 2. The operator G_σ verifies the following:

- $G_\sigma(e_0; w) = 1,$
- $G_\sigma(e_1; w) = w + \frac{1+2w}{\sigma},$
- $G_\sigma(e_2; w) = w^2 + \frac{6(\sigma+1)w^2 + 4(\sigma+2)w + 2}{(\sigma-1)\sigma}.$

Lemma 3. For the operators G_σ , we have the following:

- $G_\sigma(y - w; w) = \frac{(2w+1)}{\sigma},$

$$2. \quad G_{\sigma}((y-w)^2; w) = \frac{2((\sigma+5)w(w+1)+1)}{(\sigma-1)\sigma}.$$

Lemma 4. For $s \in \mathbb{N} \cup \{0\}$, if we define

$$U_{\sigma,s}(w) = \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) \left(\frac{l-1}{\sigma+2} - w \right)^s,$$

$$\text{then } U_{\sigma,0}(w) = 1, U_{\sigma,1}(w) = 0 \text{ and } U_{\sigma,2}(w) = \frac{w(1+w)}{\sigma+2}.$$

Further, there holds the recurrence relation

$$(\sigma+2)U_{\sigma,s+1}(w) = w(1+w)[U'_{\sigma,s}(w) + sU_{\sigma,s-1}(w)].$$

Consequently, for all $w \in [0, \infty)$

$$U_{\sigma,s}(w) = O(\sigma^{-[(s+1)/2]}),$$

where $[\alpha]$ denotes the integral part of α .

Lemma 5. [15] Let the function $T_{\sigma,s}(w)$, $s \in \mathbb{N} \cup \{0\}$, be defined as

$$\begin{aligned} T_{\sigma,s}(w) &= G_{\sigma}((y-w)^s, w) \\ &= (\sigma+1) \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) \int_0^{\infty} v_{\sigma+2,l-1}(y)(y-w)^s dy. \end{aligned}$$

$$\text{Then } T_{\sigma,0}(w) = 1, T_{\sigma,1}(w) = \frac{(2w+1)}{\sigma}, T_{\sigma,2}(w) = \frac{2((\sigma+5)w(w+1)+1)}{(\sigma-1)\sigma} \text{ and}$$

$$(\sigma-s)T_{\sigma,s+1}(w) = w(1+w)[T'_{\sigma,s}(w) + 2sT_{\sigma,s-1}(w)] + (s+1)(1+2w)T_{\sigma,s}(w)..$$

Proof. By definition, we have

$$\begin{aligned} T'_{\sigma,s}(w) &= (\sigma+1) \sum_{l=1}^{\infty} b'_{\sigma+2,l-1}(w) \int_0^{\infty} v_{\sigma+2,l-1}(y)(y-w)^s dy \\ &\quad + s(\sigma+1) \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) \int_0^{\infty} v_{\sigma+2,l-1}(y)(y-w)^{s-1} dy. \end{aligned}$$

Using the identities

$$b'_{\sigma+2,l-1}(w) = \frac{l-1-(\sigma+2)w}{w(1+w)} v_{\sigma+2,l-1}(w)$$

we have

$$\begin{aligned}
 & w(1+w)[T'_{\sigma,s}(w) + sT_{\sigma,s-1}(w)] \\
 = & (\sigma+1) \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) \int_0^{\infty} [(l-1) - (\sigma+2)y + (\sigma+2)(y-w)] v_{\sigma+2,l-1}(y)(y-w)^s dy \\
 = & (\sigma+1) \sum_{l=1}^{\infty} \int_0^{\infty} (1+y) b'_{\sigma+2,l-1}(y)(y-w)^s dy + (\sigma+2)T_{\sigma,s+1}(w) \\
 = & -(s+1)(1+2w)T_{\sigma,s}(w) - sw(1+w)T_{\sigma,s}(w) \\
 & -(s+2)T_{\sigma,s+1}(w) + (\sigma+2)T_{\sigma,s+1}(w).
 \end{aligned}$$

Thus by rearrangement of terms, we obtain

$$(\sigma-s)T_{\sigma,s+1}(w) = w(1+w)[T'_{\sigma,s}(w) + 2sT_{\sigma,s-1}(w)] + (s+1)(1+2w)T_{\sigma,s}(w).$$

This completes the proof of recurrence relation. From the above recurrence relation, it is again verified for all $w \in [0, \infty)$ that

$$T_{\sigma,s}(w) = O(\sigma^{-[s+1]/2}).$$

Lemma 6. It is easily verified from Lemma 4 that for each $w \in (0, \infty)$

$$G_{\sigma}(y^i; w) = \frac{(\sigma+i+1)!(\sigma-i)!}{(\sigma+1)!\sigma!} w^i + i^2 \frac{(\sigma+i)!(\sigma-i)!}{(\sigma+1)!\sigma!} w^{i-1} + O\left(\frac{1}{\sigma^2}\right).$$

Lemma 7. There exist the polynomial $Q_{i,j,r}(w)$ independent of σ and l such that

$$\begin{aligned}
 & (w(1+w))^r \frac{d}{dw} [v_{\sigma+2,l-1}(w)] \\
 = & \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (\sigma+2)^i (l-1 - (\sigma+2)w)^j Q_{i,j,r}(w) v_{\sigma+2,l-1}(w).
 \end{aligned}$$

By C_0 , we denote the class of continuous functions on the interval $(0, \infty)$ having a compact support and C_0^r is the class of r times continuously differentiable functions with $C_0^r \subset C_0$. The function f is said to belong to the generalized Zygmund class $Liz(\alpha, 1, a, b)$, if there exists a constant M such that $\omega_2(f, \delta) \leq M\delta^\alpha$, $\delta > 0$, where $\omega_2(f, \delta)$ denotes the modulus of continuity of second order on the interval $[a, b]$. The class $Liz(\alpha, 1, a, b)$ is more commonly denoted by $lip^*(\alpha, a, b)$.

We assume throughout the paper that $0 < a' < a'' < b'' < b' < b < \infty$.

$G^r = \{g: g \in C_0^{r+2}, \text{supp } g \subset [a', b']\}$. For $0 < \xi \leq 1$ and r times continuously differentiable functions f with $\text{supp } f \subset [a', b']$, the Peetre's K -functionals is defined by

$$K_r(\xi, f) = \inf_{g \in G^{(r)}} \left[\|f^{(r)} - g^{(r)}\|_{C[a', b']} + \xi \left\{ \|g^{(r)}\|_{C[a', b']} + \|g^{(r+2)}\|_{C[a', b']} \right\} \right],$$

For $0 < \alpha < 2$, the symbol $C_0^r(\alpha, 1, a, b)$ denotes the set of functions for which

$$\sup_{0 < \xi \leq 1} \xi^{-\alpha/2} K_r(\xi, f) < C.$$

Lemma 8. Let $f^{(r)} \in C_0$ with $\text{supp } f \subset [a'', b'']$ and if $f \in C_0^r(\alpha, 1, a', b')$, we have $f^{(r)} \in Liz(\alpha, 1, a', b')$.

Proof. Let $g \in G^{(r)}$, then for $f \in C_0^r(\alpha, 1, a', b')$, we have

$$\begin{aligned} |\Delta_{\delta}^2 f^{(r)}(w)| &\leq |\Delta_{\delta}^2 (f^{(r)} - g^{(r)})(w)| + |\Delta_{\delta}^2 g^{(r)}(w)| \\ &\leq \|\Delta_{\delta}^2 (f^{(r)} - g^{(r)})\|_{C[a', b']} + \delta^2 \|g^{(r+2)}\|_C [a', b'] \\ &\leq MK_r(\delta^2, f) \\ &\leq M\delta^{\alpha}. \end{aligned}$$

□

Lemma 9. If f is r times differentiable on $[0, \infty)$, such that $f^{(r)} \in C_{\beta}[0, \infty)$ as $y \rightarrow \infty$, the for $r = 1, 2, 3, \dots$ and $\sigma > \alpha + r$ we have

$$G_{\sigma}^{(r)}(f, w) = \frac{(\sigma + r + 1)! (\sigma - r + 1)!}{(\sigma + 1)! \sigma!} \sum_{l=1}^{\infty} v_{\sigma+r+2, l-1}(w) \int_0^{\infty} v_{\sigma-r+2, l+r-1}(y) f^{(r)}(y) dy.$$

Proof. First

$$G'_{\sigma}(f, w) = \sum_{l=1}^{\infty} b'_{\sigma+2, l-1}(w) \int_0^{\infty} v_{\sigma+2, l-1}(y) f(y) dy$$

Now using the identities

$$(4) \quad b'_{\sigma+2, l-1}(w) = (\sigma + 2)[v_{\sigma+3, l-2}(w) - v_{\sigma+3, l-1}(w)],$$

for $l \geq 1$, we have

$$\begin{aligned} G'_{\sigma}(f, w) &= \sum_{l=1}^{\infty} (\sigma + 1)(\sigma + 2)[v_{\sigma+3, l-2}(w) - v_{\sigma+3, l-1}(w)] \int_0^{\infty} v_{\sigma+2, l-1}(y) f(y) dy \\ &= (\sigma + 1)(\sigma + 2) \sum_{l=1}^{\infty} [v_{\sigma+3, l-1}(w)] \int_0^{\infty} [v_{\sigma+2, l}(y) - v_{\sigma+2, l-1}(y)] f(y) dy \\ &= (\sigma + 1)(\sigma + 2) \sum_{l=1}^{\infty} [v_{\sigma+3, l-1}(w)] \int_0^{\infty} -\frac{1}{\sigma + 1} b'_{\sigma+1, l}(y) f(y) dy. \end{aligned}$$

Integrating by parts, we get

$$G'_{\sigma}(f, w) = (\sigma + 2) \sum_{l=1}^{\infty} v_{\sigma+3, l-1}(w) \int_0^{\infty} v_{\sigma+1, l}(y) f'(y) dy.$$

Thus the result is true for $r = 1$. We prove the result by induction method. Suppose that the result is true for $r = i$, then

$$G_{\sigma}^{(i)}(f, w) = \frac{(\sigma + i + 1)! (\sigma - i + 1)!}{(\sigma + 1)! \sigma!} \sum_{l=1}^{\infty} v_{\sigma+i+2, l-1}(w) \int_0^{\infty} v_{\sigma-i+2, l+i-1}(y) f^{(i)}(y) dy.$$

Thus by using the identities of equation (4), we have

$$\begin{aligned}
 & G_{\sigma}^{(i+1)}(f, w) \\
 &= \frac{(\sigma + i + 1)! (\sigma - i + 1)!}{(\sigma + 1)! \sigma!} \\
 & \times \sum_{l=1}^{\infty} (\sigma + i + 2) [v_{\sigma+i+3, l-2}(w) - v_{\sigma+i+3, l-1}(w)] \int_0^{\infty} v_{\sigma+i+2, l+i-1} f^{(i)}(y) dy \\
 &= \frac{(\sigma + i + 2)! (\sigma - i + 1)!}{(\sigma + 1)! \sigma!} \sum_{l=1}^{\infty} v_{\sigma+i+3, l-1}(w) \int_0^{\infty} [v_{\sigma-i+2, l+i}(y) - v_{\sigma-i+2, l+i-1}(y)] f^{(i)}(y) dy.
 \end{aligned}$$

Finally, integrating by parts, yields

$$G_{\sigma}^{(i+1)}(f, w) = \frac{(\sigma + i + 2)! (\sigma - i + 1)!}{(\sigma + 1)! \sigma!} \sum_{l=1}^{\infty} v_{\sigma+i+3, l-1}(w) \int_0^{\infty} v_{\sigma-i+1, l+i} f^{(i+1)}(y) dy.$$

This completes the proof of the lemma.

III. CONVERGENCE RESULTS

In this section we present the following results.

Theorem 1. Let $f \in C_B[0, \infty)$ and $f^{(r)}$ exists at a point $w \in (0, \infty)$. Then we have

$$G_{\sigma}^{(r)}(f, w) = f^{(r)}(w)$$

as $\sigma \rightarrow \infty$.

Proof. By Taylor expansion of f , we have

$$f(y) = \sum_{i=0}^r \frac{f^{(i)}(w)}{i!} (y - w)^i + \epsilon(y, w)(y - w)^r,$$

where $\epsilon(y, w) \rightarrow 0$ as $y \rightarrow w$.

Now,

$$\begin{aligned}
 G_{\sigma}^{(r)}(f, w) &= \sum_{i=0}^r \frac{f^{(i)}(w)}{i!} G_{\sigma}^{(r)}((y - w)^i, w) + G_{\sigma}^{(r)}(\epsilon(y, w)(y - w)^r, w) \\
 &= R_1 + R_2, \text{ say.}
 \end{aligned}$$

First we calculate R_1 . Using the binomial expansion of $(y - w)^i$ and using the Remark 6, we have

$$\begin{aligned}
 R_1 &= \sum_{i=0}^r \frac{f^{(i)}(w)}{i!} \sum_{v=0}^i \binom{i}{v} (-w)^{i-v} \frac{\partial^r}{\partial w^r} G_{\sigma}(y^v, w) \\
 &= \frac{f^{(r)}(w)}{r!} \frac{d^r}{dw^r} \left[\frac{(\sigma + r + 1)! (\sigma - r)!}{(\sigma + 1)! \sigma!} w^r + \text{terms containing lower power of } w \right] \\
 &= f^{(r)}(w) \left[\frac{(\sigma + r + 1)! (\sigma - r)!}{(\sigma + 1)! \sigma!} \right].
 \end{aligned}$$

Now applying Lemma 7, we obtain

$$R_2 = G_\sigma^{(r)}(\epsilon(y, w)(y - w)^r, w), \text{ so that}$$

$$|R_2| \leq \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} \sigma^i \frac{|Q_{i,j,r}(w)|}{(w(1+w))^r} \sum_{l=1}^{\infty} |l - \sigma w - 2w - 1|^j v_{\sigma+2,l-1}(w) \times$$

$$\times \int_0^{\infty} v_{\sigma+2,l-1}(y) |\epsilon(y, w)| |y - w|^r dy$$

Since, $\epsilon(y, w) \rightarrow 0$ as $y \rightarrow w$ for given $\epsilon > 0$, there exist a δ such that $|\epsilon(y, w)| < \epsilon$ whenever $0 < |y - w| < \delta$. If $\alpha \geq \max\{\beta, r\}$ where α is any integer, we can find a constant $C > 0$, $|\epsilon(y, w)(y - w)^r| \leq C|y - w|^\alpha$ for $|y - w| \geq \delta$. Therefore,

$$|R_2| \leq C \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} \sigma^i \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) |l - \sigma w - 2w - 1|^j$$

$$\times \left\{ \epsilon \int_{|y-w| < \delta} v_{\sigma+2,l-1}(y) |y - w|^r dy + \int_{|y-w| \geq \delta} v_{\sigma+2,l-1}(y) |y - w|^\alpha dy \right\}$$

$$= R_3 + R_4, \text{ say.}$$

Applying Cauchy-Schwarz inequality for integration and summation respectively, we obtain

$$R_3 \leq \epsilon C \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} \sigma^i \left\{ \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) |l - \sigma w - 2w - 1|^{2j} \right\}^{1/2} \times$$

$$\times \left\{ \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) (y - w)^{2r} dy \right\}^{1/2}.$$

Using Lemma 4 and Lemma 5, we get

$$R_3 = \epsilon \cdot O(n^{r/2}) O(n^{-r/2}) = \epsilon.$$

Again using the Cauchy-Schwarz inequality, lemma 4 it follow that

$$R_4 \leq C \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} \sigma^i \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) |l - \sigma w - 2w - 1|^j \int_{|y-w| \geq \delta} v_{\sigma+2,l-1}(y) |y - w|^\alpha dy$$

$$\leq C \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} \sigma^i \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) |l - \sigma w - 2w - 1|^j \left\{ \int_{|y-w| \geq \delta} v_{\sigma+2,l-1}(y) dy \right\}^{1/2}$$

$$\times \left\{ \int_{|y-w| \geq \delta} v_{\sigma+2,l-1}(y) (y - w)^{2\alpha} dy \right\}^{1/2}$$

$$\leq C \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} \sigma^i \left\{ \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) |l - \sigma w - 2w - 1|^{2j} \right\}^{1/2}$$

$$\times \left\{ \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) \int_0^{\infty} v_{\sigma+2,l-1}(y) (y - w)^{2\alpha} dy \right\}^{1/2}$$

$$= \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} \sigma^i O(\sigma^{1/2}) O(\sigma^{-\alpha/2}) = O(\sigma^{(r-\alpha)/2}) = o(1).$$

Collecting the estimates R_1, R_2, R_3 and R_4 , we obtain the required result. $\square$

Theorem 2. Let $f \in C_\beta[0, \infty)$. If $f^{(r+2)}$ exists at a point $w \in (0, \infty)$. Then

$$\lim_{\sigma \rightarrow \infty} \sigma [G_\sigma^{(r)}(f, w) - f^{(r)}(w)] = r(r+1)\sigma f^{(r)}(w) + (r+1)(1+2w)f^{(r+1)}(w)$$

$$+ x(1+x)f^{(r+2)}(w).$$

Proof. Using Taylor's expansion of f , we have

$$f(y) = \sum_{i=0}^{r+2} \frac{f^{(i)}(w)}{i!} (y-w)^i + \epsilon(y, w)(y-w)^{r+2},$$

where $\epsilon(y, w) \rightarrow 0$ as $y \rightarrow w$ and $\epsilon(y, w) = O((y-w)^\beta)$, $y \rightarrow \infty$ for $\beta > 0$. Applying Lemma 5, we have

$$\begin{aligned} & \sigma [G_\sigma^{(r)}(f, w) - f^{(r)}(w)] \\ &= \sigma \left[\sum_{i=0}^{r+2} \frac{f^{(i)}(w)}{i!} G_\sigma^{(r)}((y-w)^i, w) - f^{(r)}(w) \right] \\ & \quad + \sigma G_\sigma^{(r)}(\epsilon(y, w)(y-w)^{r+2}, w) \\ &= F_1 + F_2, \text{ say.} \end{aligned}$$

First, we have

$$\begin{aligned} F_1 &= \sigma \sum_{i=0}^{r+2} \frac{f^{(i)}(w)}{i!} \sum_{j=0}^i \binom{i}{j} (-w)^{i-j} G_\sigma^{(r)}(y, w) y^j dy - \sigma f^{(r)}(w) \\ &= \sigma \frac{f^{(r)}(w)}{r!} [G_\sigma^r(y^r, w) - (r!)] \\ & \quad + \sigma \frac{f^{(r+1)}(w)}{(r+1)!} [(r+1)(-w)G_\sigma^r(y^r, w) + G_\sigma^r(y^{(r+1)}, w)] \\ & \quad + \sigma \frac{f^{(r+2)}(w)}{(r+2)!} \left[\frac{(r+2)(r+1)}{2!} w^2 G_\sigma^r(y^r, w) + (r+2)(-w)G_\sigma^r(y^{(r+1)}, w) \right. \\ & \quad \left. + G_\sigma^r(y^{(r+2)}, w) \right]. \end{aligned}$$

Therefore, by 6, we have

$$\begin{aligned} F_1 &= \sigma f^{(r)}(w) \left[\frac{(\sigma+r+1)!(\sigma-r)!}{(\sigma+1)!\sigma!} (r)! - (r)! \right] \\ & \quad + \sigma \frac{f^{(r+1)}(w)}{(r+1)!} [(r+1)(-w) \left\{ \frac{(\sigma+r+1)!(\sigma-r)!}{(\sigma+1)!\sigma!} (r)! \right\} \\ & \quad + \left\{ \frac{(\sigma+r+2)!(\sigma-r-1)!}{(\sigma+1)!\sigma!} (r+1)!(w) + (r+1)^2 \frac{(\sigma+r+1)!(\sigma-r-1)!}{(\sigma+1)!\sigma!} (r)! \right\}] \\ & \quad + \sigma \frac{f^{(r+2)}(w)}{(r+2)!} \left[(r+2)(r+1)w^2 \frac{(\sigma+r+1)!(\sigma-r)!}{(\sigma+1)!\sigma!} (r)! \right. \\ & \quad + (r+2)(-w) \left\{ \frac{(\sigma+r+2)!(\sigma-r-1)!}{(\sigma+1)!\sigma!} (r+1)!w + (r+1)^2 \frac{(\sigma+r+1)!(\sigma-r-1)!}{(\sigma+1)!\sigma!} r! \right\} \\ & \quad + \left\{ \frac{(\sigma+r+3)!(\sigma-r-2)!}{(\sigma+1)!\sigma!} \frac{(r+2)!}{2} w^2 + (r+2)^2 \frac{(\sigma+r+2)!(\sigma-r-2)!}{(\sigma+1)!\sigma!} (r+1)!w \right\} \\ & \quad \left. + O(\sigma^{(-2)}) \right] \end{aligned}$$

In order to complete the proof of the theorem it is sufficient to show that $F_2 \rightarrow 0$ as $\sigma \rightarrow \infty$ which easily follows proceeding along the lines of the proof of Theorem 1 and by using Lemma 4, Lemma 5 and Lemma 7.

Theorem 3. Let $[a_1, b_1] \subset (a, b) \subset (0, \infty)$, and $f^{(r)}$ exists and is continuous on $[a, b]$ then for sufficiently large σ ,

$$\|G_\sigma^{(r)}(f, \bullet) - f^{(r)}\|_{[a_1, b_1]} \leq C((\sigma)^{-1} \|f\| + \omega_2(f^{(r)}; (\sigma)^{-1/2}; [a_1, b_1])),$$

where C is independent of f and σ .

Proof. We have that

$$\begin{aligned} \|G_{\sigma}^{(r)}(f) - f^{(r)}\|_{[a_1, b_1]} &\leq \|G_{\sigma}^{(r)}(f - f_{\lambda, 2}, \bullet)\|_{[a_1, b_1]} + \|G_{\sigma}^{(r)}(f_{\lambda, 2}, \bullet) - f_{\lambda, 2}^{(r)}\|_{[a_1, b_1]} \\ &\quad + \|f^{(r)} - f_{\lambda, 2}^{(r)}\|_{[a_1, b_1]} \\ &= A_1 + A_2 + A_3, \text{ say.} \end{aligned}$$

Since, $(f_{\lambda, 2})^{(r)} = (f^{(r)})_{\lambda, 2}$, by property of the Steklov mean it follows that

$$A_3 \leq C \omega_2(f^{(r)}, \lambda, [a_1, b_1]).$$

Next, applying Theorem 2 and the Riesz-Thorin interpolation theorem, we have that

$$\begin{aligned} A_2 &\leq C(\sigma)^{-1} \sum_{m=r}^{r+2} \|f_{\lambda, 2}^{(m)}\|_{[a_1, b_1]} \\ &\leq C(\sigma)^{-1} (\|f_{\lambda, 2}\|_{[a_1, b_1]} + \|f_{\lambda, 2}^{(r+2)}\|_{[a_1, b_1]}) \\ &\leq C(\sigma)^{-1} (\|f_{\lambda, 2}\|_{[a_1, b_1]} + \|(f^{(r)})''_{\lambda, 2}\|_{[a_1, b_1]}). \\ A_2 &\leq C(\sigma)^{-1} \sum_{m=r}^{r+2} \|f_{\lambda, 2}^{(m)}\|_{[a_1, b_1]} \\ &\leq C(\sigma)^{-1} (\|f_{\lambda, 2}\|_{[a_1, b_1]} + \|f_{\lambda, 2}^{(r+2)}\|_{[a_1, b_1]}) \\ &\leq C(\sigma)^{-1} (\|f_{\lambda, 2}\|_{[a_1, b_1]} + \|(f^{(r)})''_{\lambda, 2}\|_{[a_1, b_1]}). \end{aligned}$$

Hence, by properties of Steklov mean we have

$$A_2 \leq C(\sigma)^{-1} (\|f\| + \lambda^{-2} \omega_2(f^{(r)}, \lambda, [a_1, b_1])).$$

In order to find estimation for A_3 , let us write $f - f_{\lambda, 2} = \Psi$. By our hypothesis, we can write

$$\begin{aligned} \Psi(y) &= \sum_{m=0}^r \frac{\Psi^{(m)}(w)}{m!} (y - w)^m + \frac{\Psi^{(r)}(\xi) - \Psi^{(r)}(w)}{r!} (y - w)^r \chi(y) \\ &\quad + \lambda(y, w)(1 - \chi(y)), \end{aligned}$$

where ξ lies between y and w , and χ is the characteristic function of the interval (a, b) . For $y \in (a, b)$ and $w \in [a_1, b_1]$, we have

$$\Psi(y) = \sum_{m=0}^r \frac{\Psi^{(m)}(w)}{m!} (y - w)^m + \frac{\Psi^{(r)}(\xi) - \Psi^{(r)}(w)}{r!} (y - w)^r,$$

If $y \in [0, \infty) \setminus (a, b)$, $w \in [a_1, b_1]$ then, we define

$$\lambda(y, w) = \Psi(y) - \sum_{m=0}^r \frac{\Psi^{(m)}(w)}{m!} (y - w)^m.$$

Now,

$$\begin{aligned} G_{\sigma}(\Psi(y), w) &= \sum_{m=0}^r \frac{\Psi^{(m)}(w)}{m!} G_{\sigma}^{(r)}((y-w)^m, w) \\ &+ \frac{G_{\sigma}^{(r)}\left(\left(\Psi^{(r)}(\xi) - \Psi^{(r)}(w)\right)(y-w)^r \chi(y), w\right)}{r!} \\ &+ G_{\sigma}^{(r)}(\lambda(y, w)(1 - \chi(y)), w) \\ &= \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3, \text{ say.} \end{aligned}$$

By using the identity, we write that

$$\begin{aligned} \mathcal{B}_1 &= \sum_{m=0}^r \frac{\Psi^{(m)}(w)}{m!} G_{\sigma}^{(r)}((y-w)^m, w) \\ &= \sum_{m=0}^r \frac{\Psi^{(m)}(w)}{m!} \sum_{l=0}^m \binom{m}{l} (-w)^{m-l} G_{\sigma}^{(r)}(y^l, w) \\ &= \lambda_{\sigma}(c, r) \Psi^{(r)}(w). \end{aligned}$$

Hence,

$$|\mathcal{B}_1| \leq C \lambda_{\sigma}(c, r) \|f^{(r)} - f_{\lambda, 2}^{(r)}\|_{[a_1, b_1]}.$$

Next, using Lemma 7 and Cauchy-Schwarz inequality for integration and then for summation we get

$$\begin{aligned} |\mathcal{B}_2| &\leq \frac{2}{r!} \|f^{(r)} - f_{\lambda, 2}^{(r)}\|_{[a_1, b_1]} G_{\sigma}^{(r)}(\chi(y)|y-w|^r, w) \\ &\leq \frac{2}{r!} \|f^{(r)} - f_{\lambda, 2}^{(r)}\|_{[a_1, b_1]} \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} (\sigma+2)^i \frac{|Q_{i, j, r}(w)|}{(w(1+w))^r} \sum_{l=1}^{\infty} v_{\sigma+2, l-1}(w) |l-1-(\sigma+2)w|^j \times \\ &\quad \times \int_0^{\infty} v_{\sigma+2, l-1}(y) \chi(y) |y-w|^r dy \\ &\leq \frac{2}{r!} \|f^{(r)} - f_{\lambda, 2}^{(r)}\|_{[a_1, b_1]} \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} (\sigma+2)^i \frac{|Q_{i, j, r}(w)|}{(w(1+w))^r} \sum_{l=1}^{\infty} v_{\sigma+2, l-1}(w) |l-1-(\sigma+2)w|^j \times \\ &\quad \times \left(\int_0^{\infty} v_{\sigma+2, l-1}(y) dt \right)^{1/2} \left(\int_0^{\infty} v_{\sigma+2, l-1}(y) (y-w)^{2r} dt \right)^{1/2} \\ &\leq C \|f^{(r)} - f_{\lambda, 2}^{(r)}\|_{[a_1, b_1]} \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} (\sigma+2)^i \left(\sum_{l=1}^{\infty} v_{\sigma+2, l-1}(w) (l-1-(\sigma+2)w)^{2j} \right)^{1/2} \times \\ &\quad \times ((\sigma+2) \sum_{l=1}^{\infty} v_{\sigma+2, l-1}(w) \int_0^{\infty} v_{\sigma+2, l-1}(y) (y-w)^{2r} dy)^{1/2} \\ &\leq C \|f^{(r)} - f_{\lambda, 2}^{(r)}\|_{[a_1, b_1]} \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} (\sigma+2)^i O(\sigma^{j/2}) O(\sigma^{-r/2}) \\ &\leq C \|f^{(r)} - f_{\lambda, 2}^{(r)}\|_{[a_1, b_1]}. \end{aligned}$$

Since $y \in (0, \infty) \setminus (a, b)$, we can choose a $\delta > 0$ in such a way that $|y - w| \geq \delta$ for all $w \in [a_1, b_1]$. Thus, by Lemma 7, we obtain

$$|B_3| \leq \sum_{\substack{2l+j \leq r \\ i,j \geq 0}} (\sigma + 2)^i \frac{|Q_{i,j,r}(w)|}{w^r(1+w)^r} (\sigma + 2) \sum_{l=1}^{\infty} v_{\sigma+2,l-1}(w) |1 - (\sigma + 2)w|^i \int_{|y-w| \geq \delta} v_{\sigma+2,l-1}(y) |\lambda(y, w)| dy$$

For $|y - w| \geq \delta$, we can find a constant $C > 0$ such that $|\lambda(y, w)| \leq C|y - w|^s$. Finally using Cauchy-Schwarz inequality for integration, it easily follows that $B_3 = O((\sigma)^{-s})$ for any $s > 0$.

Combining the estimates $B_1 - B_3$, we obtain in view of (c) of Steklov mean,

$$\begin{aligned} A_3 &\leq C \|f^{(r)} - f_{\lambda, 2l+2}^{(r)}\|_{[a_1, b_1]} + O((\sigma)^{-s}) \\ &\leq C \omega_{2l+2}(f^{(r)}, \lambda, [a_1, b_1]) + O((\sigma)^{-s}). \end{aligned}$$

Taking $\lambda = (\sigma)^{-1/2}$ the proof is completed.

Our last result proves local rate of approximation.

For simultaneous approximation properties of the operator $G_{\sigma}(f, w)$, we use the transformed operator $\widehat{G}_{\sigma}(f, r, w)$. This operator is defined by

$$\widehat{G}_{\sigma,r}(f, w) = \frac{(\sigma + r + 1)! (\sigma - r + 1)!}{(\sigma + 1)! \sigma!} \sum_{l=1}^{\infty} v_{\sigma+r+2,l-1}(w) \int_0^{\infty} v_{\sigma-r+2,l+r-1}(y) f(y) dy,$$

and

$$\widehat{G}_{\sigma,r}(1, w) = \frac{1}{\sigma - r + 1}$$

Clearly

$$\widehat{G}_{\sigma}(f^{(r)}, w) = \widehat{G}_{\sigma}^{(r)}(f, w).$$

Theorem 4. Let $f^{(r)} \in C_B[0, \infty)$. Then

$$\begin{aligned} &|\widehat{G}_{\sigma,r}(f, w) - f^{(r)}(w)| \\ &\leq \omega \left(f^{(r)}, \left(\frac{(\sigma + r + 1)! (\sigma - r + 1)!}{(\sigma + 1)! \sigma!} (\sigma + 1)(\sigma + 2) + 2(\sigma + 1)(2\sigma + 5)w(1 + w) \right)^{1/2} \right) + \\ &+ \frac{(\sigma - r)}{(\sigma - r + 1)} |f^{(r)}(w)|. \end{aligned}$$

Proof. We have

$$G_{\sigma,r}(f^{(r)}, w) = \frac{(\sigma + r + 1)! (\sigma - r + 1)!}{(\sigma + 1)! \sigma!} \sum_{l=1}^{\infty} v_{\sigma+r+2,l-1}(w) \int_0^{\infty} v_{\sigma-r+2,l+r-1}(y) f^{(r)}(y) dy,$$

Since

$$\sum_{l=1}^{\infty} v_{\sigma+r+2,l-1}(w) = 1$$

and

$$\frac{1}{\widehat{G}_{\sigma,r}(1,w)} \left(\frac{(\sigma+r+1)! (\sigma-r+1)!}{(\sigma+1)! \sigma!} \int_0^\infty v_{\sigma-r+2,l+r-1}(y) dy \right) = 1.$$

Therefore

$$\begin{aligned} & |G_{\sigma,r}(f^{(r)}, w) - f^{(r)}(x)| \\ &= \left| \frac{(\sigma+r+1)! (\sigma-r+1)!}{(\sigma+1)! \sigma!} \sum_{l=1}^\infty v_{\sigma+r+2,l-1}(w) \int_0^\infty v_{\sigma-r+2,l+r-1}(y) f^{(r)}(y) dy \right. \\ &\quad \left. - f^{(r)} \frac{\widehat{G}_{\sigma,r}(1,w)}{\widehat{G}_{\sigma,r}(1,w)} \right| \\ &\leq \left| \frac{(\sigma+r+1)! (\sigma-r+1)!}{(\sigma+1)! \sigma!} \sum_{l=1}^\infty v_{\sigma+r+2,l-1}(w) \int_0^\infty v_{\sigma-r+2,l+r-1}(y) (f^{(r)}(y) - f^{(r)}(w)) dy \right| \\ &\quad + \frac{(\sigma-r)}{(\sigma-r+1)} |f^{(r)}(w)| \\ &= \widehat{G}_{\sigma,r}(|f^{(r)}(y) - f^{(r)}(w)|, w) + \frac{(\sigma-r)}{(\sigma-r+1)} |f^{(r)}(w)|. \\ &= \mathcal{F}_1 + \mathcal{F}_2, \text{ say.} \end{aligned}$$

Now for $\mathcal{F}_1$ in view of the inequality

$$|f^{(r)}(y) - f^{(r)}(w)| \leq \left(1 + \frac{|y-w|}{\delta} \right) \omega(f^{(r)}; \delta),$$

we have

$$|\widehat{G}_{\sigma,r}(f^{(r)}, w) - f^{(r)}(w)| \leq \omega(f^{(r)}; \delta) \left(\widehat{G}_{\sigma,r}(1, w) + \frac{\widehat{G}_{\sigma,r}(|y-w|, w)}{\delta} \right).$$

Using Cauchy-Schwarz inequality, we obtain

$$\widehat{G}_{\sigma,r}(|y-w|, w) \leq (\widehat{G}_{\sigma,r}((y-w)^2), w)^{\frac{1}{2}}.$$

By direct calculations, we have

$$\widehat{G}_{\sigma,r}((y-w)^2, w) = \frac{(\sigma+r+1)! (\sigma-r)}{(\sigma+1)! \sigma!} ((\sigma+1)(\sigma+2) + 2(\sigma+1)(2\sigma+5)w(1+w))$$

Hence,

$$|G_{\sigma,r}(f^{(r)}, w) - f^{(r)}(w)| \leq \omega(f^{(r)}; \delta) \left(\widehat{G}_{\sigma,r}(1, w) + \frac{\widehat{G}_{\sigma,r}(|y-w|, w)}{\delta} \right) + \frac{(\sigma-r)}{(\sigma-r+1)} |f^{(r)}(w)|.$$

The theorem now follows by choosing

$$\delta = \left(\frac{(\sigma+r+1)! (\sigma-r)}{(\sigma+1)! \sigma!} ((\sigma+1)(\sigma+2) + 2(\sigma+1)(2\sigma+5)w(1+w)) \right)^{1/2}.$$

IV. CONCLUSION

This study provides the study of behaviour of derivatives of the summation-integral type operator $G_\sigma(f, w)$ and concludes that the operator sequence $\{G_\sigma^{(r)}(f, w)\}_{\sigma=1}^\infty$ converges to the function $f^{(r)}(x)$. A Voronoskaja type asymptotic formula for r -th derivative of the operator was also obtained with an improved rate of approximation $\left(\frac{1}{n}\right)$ for sufficient smooth functions. Using the properties of Steklov means error estimates are calculated in terms of modulus of smoothness. The approach can be applied to other summation-integral type operators like Stancu variant of Szász–Mirakjan–Baskakov operator to study simultaneous approximation.

V. REFERENCES

- [1] J.L. Durrmeyer, "Une formule d'inversion de la transformée de Laplace: Applications à la théorie des moments," *PhD thesis*, 1967.
- [2] P.N. Agrawal, V.Gupta, "Simultaneous approximation by linear combination of modified Bernstein polynomials," 1989.
- [3] Z. Ditzian, K. Ivanov, "Bernstein-type operators and their derivatives," *Approx. Theory*, vol. 56(1), pp. 72-90, 1989.
- [4] V. Gupta, P.N. Agrawal, "Lp-approximation by iterative combination of Phillips operators," *Publ. Inst. Math. (Beograd) (N.S.)*, pp. 52:67-76, 1992.
- [5] V. Gupta, P.N. Agrawal, A. Sahai, and T.A.K. Sinha, "Lp-approximation by combination of modified Szasz–Mirakyan operators," *Demonstr. Math.*, vol. 23(3), pp. 577-592, 1990.
- [6] A. Sahai, G. Prasad, "On simultaneous approximation by modified Lupas operators," *Approx. Theory*, pp. 45(2):122-128, 1985.
- [7] S.P. Singh, "On approximation by modified Szász operators," *Math. Chron.*, p. 15:39–48, 1986.
- [8] V. Gupta and G. Srivastava, "Approximation by Durrmeyer-type operators," *In Ann. Polon. Math.*, vol. 64, pp. 153-159, 1996.
- [9] H. Karsli, E. Ibikli, "Rate of convergence of Chlodowsky-type Bernstein operators for functions of bounded variation," *Numer. Funct. Anal. Optim.*, pp. 28(3-4):367–378, 2007.
- [10] U. Abel, M. Ivan, Z. Li., "Local approximation by generalized Baskakov–Durrmeyer operators," *Numer. Funct. Anal. Optim*, pp. 28(3-4):245–264, 2007.
- [11] S. Bajpeyi, A.S. Kumar, and I. Mantellini, "Approximation by Durrmeyer type exponential sampling operators," *Numer. Funct. Anal. Optim.*, p. 43(1):16–34, 2022.
- [12] Z. Ditzian and V. Totik, "Moduli of Smoothness," *Springer Science & Business Media*, vol. 9, 2012.
- [13] E. Hewitt and K. Stromberg, "Real and Abstract Analysis: A Modern Treatment of the Theory of Functions of a Real Variable," *Springer-Verlag, New York, 1st edition*, 1965.

- [14] A. F. Timan, "Theory of Approximation of Functions of a Real Variable. Elsevier," 2014.
- [15] V. Gupta, "A note on modified Baskakov type operators," *Approx. Theory Appl.*, p. 10(3):74–78, 1994.



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